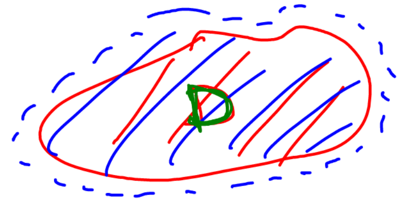
Divergence Theorem



Idea: Generalize Green's Theorem to 3-space.

Recall (Green's Theorem): Let $D \subseteq \mathbb{R}^2$ have ∂D a simple, closed, piecewise-smooth curve. If P and Q are functions with continuous partial derivatives on an open region containing D, then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA. \leftarrow$$

We first rewrite Green's using vector fields in $\mathbb{R}^2$. Letting $\vec{r}(t) = \langle x(t), y(t) \rangle$ parameterize ∂D piecewise-smoothly on $[a,b]$, we recall $\vec{N}(t) = \frac{1}{|\vec{r}\,'(t)|}\langle y'(t), -x'(t) \rangle$ is the normal curve to $\vec{r}(t)$

(i.e. $\vec{r}\,'(t) \cdot N(t) = 0$ for all t, so $\vec{n}(t)$ is orthogonal to $\vec{r}\,'(t)$ at all t).

Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$. Thus we compute:

$$\int_{\partial D} \vec{F} \cdot \vec{N}\, ds = \int_{t=a}^{b} \left(\vec{F}(\vec{r}(t)) \cdot \frac{1}{|\vec{r}\,'(t)|} \langle y'(t), -x'(t) \rangle \right) |\vec{r}\,'(t)|\, dt$$

$$= \int_{t=a}^{b} \langle P(x(t), y(t)), Q(x(t), y(t)) \rangle \cdot \langle y'(t), -x'(t) \rangle\, dt$$

$$= \int_{t=a}^{b} \left(P(x(t), y(t))\, y'(t) - Q(x(t), y(t))\, x'(t) \right) dt$$

$$= \int_{\partial D} -Q\,dx + P\,dy = \iint_D \left(\frac{\partial P}{\partial x} - -\frac{\partial Q}{\partial y} \right) dA$$

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_D \operatorname{div}(\vec{F})\, dA$$

☺

Prop (Divergence Theorem): Let $R \subseteq \mathbb{R}^3$ be a compact solid with ∂R a piecewise-smooth surface. If $\vec{F}$ is a vector field on $\mathbb{R}^3$ with continuous partial derivatives on an open region containing R, then $\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$.

We'll prove a special case. The full result follows from our special case, but the details are somewhat messy. I'll remark more on this after the proof.



pf: Assume R admits a parameterization $\vec{\alpha}(s,t,u)$ satisfying

① $\operatorname{dom}(\alpha) = [a_s, b_s] \times [a_t, b_t] \times [a_u, b_u]$, and

② ∂R is parameterized by restricting α to $\partial \operatorname{dom}(\alpha)$ s.t. $\vec{S} : D \to \partial R$ is a composition $D \longrightarrow \partial \operatorname{dom}(\alpha) \longrightarrow \partial R$

First we rewrite the surface integral as a sum:

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iint_D \langle F_x, F_y, F_z \rangle \cdot (\vec{S}_u \times \vec{S}_v)\, dA$$

$$= \iint_D F_x \vec{\imath} \cdot (\vec{S}_u \times \vec{S}_v)\, dA + \iint_D F_y \vec{\jmath} \cdot (\vec{S}_u \times \vec{S}_v)\, dA + \iint_D F_z \vec{k} \cdot (\vec{S}_u \times \vec{S}_v)\, dA$$

$$= \iint_{\partial R} F_x \vec{e}_x \cdot d\vec{S} + \iint_{\partial R} F_y \vec{e}_y \cdot d\vec{S} + \iint_{\partial R} F_z \vec{e}_z \cdot d\vec{S}.$$

We now show that $\iint_{\partial R} F_w \vec{e}_w \cdot d\vec{S} = \iiint_R \frac{\partial F_w}{\partial w}\, dV$ for $w = x, y, z$ provided α is the identity map. We'll focus on the case $w = z$, noting the others are very similar.

Let $A = [a_x, b_x] \times [a_y, b_y]$ and compute as follows:

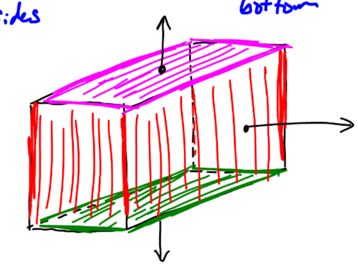
$$\iiint_R \frac{\partial F_z}{\partial z}\, dV = \iint_A \left(\int_{z=a_z}^{b_z} \frac{\partial F_z}{\partial z}\, dz \right) dA$$

$$= \iint_A F_z \Big|_{z=a_z}^{b_z} dA = \iint_A \left(F_z(x,y,b_z) - F_z(x,y,a_z) \right) dA$$

On the other hand, $\partial R = \partial \operatorname{dom}(\alpha) = T \cup S \cup B$.

(T: Top, S: sides, B: bottom)

Note that the natural orientation on B disagrees with the outward normal (so we will negate the corresponding integral), and the outward normal on S is orthogonal to $\vec{e}_z = \vec{k}$.



Hence we compute the surface integral as follows:

$$\iint_{\partial R} F_z \vec{e}_z \cdot d\vec{S} = \iint_T F_z \vec{e}_z \cdot d\vec{S} + \iint_S F_z \vec{e}_z \cdot d\vec{S} + \iint_{-B} F_z \vec{e}_z \cdot d\vec{S}$$

$$= \iint_T F_z \vec{e}_z \cdot (\vec{S}_x \times \vec{S}_y)\, dA + \iint_E F_z \vec{e}_z \cdot (\vec{S}_x \times \vec{S}_y)\, dA - \iint_B F_z \vec{e}_z \cdot (\vec{S}_x \times \vec{S}_y)\, dA$$

$$= \iint_A F_z(x,y,b_z)\, \vec{k} \cdot \vec{k}\, dA + 0 - \iint_A F_z(x,y,a_z)\, \vec{k} \cdot \vec{k}\, dA$$

$$= \iint_A \left(F_z(x,y,b_z) - F_z(x,y,a_z) \right) dA$$

Hence we have the result in this special subcase.

To see what remains, we will employ our change of α variable and use the Jacobian. Now we let

$\partial \operatorname{dom}(\alpha) = T \cup S \cup B$ as before and define a vector field $\vec{G}$ on 3-space in tuv-coordinates by the formula

$$\vec{G} = \langle \vec{F} \cdot (\vec{\alpha}_u \times \vec{\alpha}_v),\ \vec{F} \cdot (\vec{\alpha}_v \times \vec{\alpha}_t),\ \vec{F} \cdot (\vec{\alpha}_t \times \vec{\alpha}_u) \rangle$$

We already showed the special case for rectangular regions, so

$$\iiint_{\operatorname{dom}(\alpha)} \operatorname{div}(\vec{G})\, dV = \iint_{\partial \operatorname{dom}(\alpha)} \vec{G} \cdot d\vec{S}.$$

On the other hand, our assumption that $\partial \operatorname{dom}(\alpha)$ parameterizes ∂R implies for each side E of $\partial \operatorname{dom}(\alpha)$:

$$\iint_E \vec{G} \cdot d\vec{S} = \iint_E \vec{G} \cdot \vec{n}_E\, dS = \iint_E \vec{F} \cdot \vec{n}_{\alpha(E)}\, dS = \iint_{\alpha(E)} \vec{F} \cdot d\vec{S},$$

as $\vec{n}_E$ is $\pm\vec{i}$, $\pm\vec{j}$, or $\pm\vec{k}$. Thus $\displaystyle\iint_{\partial \operatorname{dom}(\alpha)} \vec{G} \cdot d\vec{S} = \iint_{\partial R} \vec{F} \cdot d\vec{S}$.

Finally we show $\displaystyle\iiint_{\operatorname{dom}(\alpha)} \operatorname{div}(\vec{G})\, dV = \iiint_R \operatorname{div}(\vec{F})\, dV$ by verifying that $\operatorname{div}(\vec{G}) = |\partial \alpha| \operatorname{div}(\vec{F})$ where $\partial \alpha$ denotes the Jacobian of the transformation α via a calculation:

$$\operatorname{div}(\vec{G}) = \frac{\partial}{\partial t}\left[\vec{F} \cdot (\alpha_u \times \alpha_v)\right] + \frac{\partial}{\partial u}\left[\vec{F} \cdot (\alpha_v \times \alpha_t)\right] + \frac{\partial}{\partial v}\left[\vec{F} \cdot (\alpha_t \times \alpha_u)\right]$$

$$\begin{aligned} = \ & \frac{\partial \vec{F}}{\partial t} \cdot (\alpha_u \times \alpha_v) + \vec{F} \cdot \frac{\partial}{\partial t}[\alpha_u \times \alpha_v] \\ & + \frac{\partial \vec{F}}{\partial u} \cdot (\alpha_v \times \alpha_t) + \vec{F} \cdot \frac{\partial}{\partial u}[\alpha_v \times \alpha_t] \\ & + \frac{\partial \vec{F}}{\partial v} \cdot (\alpha_t \times \alpha_u) + \vec{F} \cdot \frac{\partial}{\partial v}[\alpha_t \times \alpha_u] \end{aligned}$$

$$= \vec{F}_t \cdot (\alpha_u \times \alpha_v) + \vec{F} \cdot (\alpha_{ut} \times \alpha_v + \alpha_u \times \alpha_{vt})$$

$$+ \vec{F}_u \cdot (\alpha_v \times \alpha_t) + \vec{F} \cdot (\alpha_{vu} \times \alpha_t + \alpha_v \times \alpha_{tu})$$

$$+ \vec{F}_v \cdot (\alpha_t \times \alpha_u) + \vec{F} \cdot (\alpha_{tv} \times \alpha_u + \alpha_t \times \alpha_{uv})$$

$$= \underbrace{\vec{F}_t \cdot (\alpha_u \times \alpha_v) + \vec{F}_u \cdot (\alpha_v \times \alpha_t) + \vec{F}_v \cdot (\alpha_t \times \alpha_u)}$$

$$+ \vec{F} \cdot (\alpha_{tu} \times \alpha_v - \alpha_{tv} \times \alpha_u + \alpha_{uv} \times \alpha_t - \alpha_{tu} \times \alpha_v + \alpha_{tv} \times \alpha_u - \alpha_{uv} \times \alpha_t)$$

$$= \vec{F}_t \cdot (\alpha_u \times \alpha_v) + \vec{F}_u \cdot (\alpha_v \times \alpha_t) + \vec{F}_v \cdot (\alpha_t \times \alpha_u)$$

Chain rule

$$= \langle \nabla F_x \cdot \alpha_t , \nabla F_y \cdot \alpha_t , \nabla F_z \cdot \alpha_t \rangle \cdot (\alpha_u \times \alpha_v)$$

$$+ \langle \nabla F_x \cdot \alpha_u , \nabla F_y \cdot \alpha_u , \nabla F_z \cdot \alpha_u \rangle \cdot (\alpha_v \times \alpha_t)$$

$$+ \langle \nabla F_x \cdot \alpha_v , \nabla F_y \cdot \alpha_v , \nabla F_z \cdot \alpha_v \rangle \cdot (\alpha_t \times \alpha_u)$$

evaluate dot product

$$= \nabla F_x \cdot \alpha_t \,(y_u z_v - y_v z_u) - \nabla F_y \cdot \alpha_t \,(x_u z_v - x_v z_u) + \nabla F_z \cdot \alpha_t \,(x_u y_v - x_v y_u)$$

$$+ \nabla F_x \cdot \alpha_u \,(y_v z_t - y_t z_v) - \nabla F_y \cdot \alpha_u \,(x_v z_t - x_t z_v) + \nabla F_z \cdot \alpha_u \,(x_v y_t - x_t y_v)$$

$$+ \nabla F_x \cdot \alpha_v \,(y_t z_u - y_u z_t) - \nabla F_y \cdot \alpha_v \,(x_t z_u - x_u z_t) + \nabla F_z \cdot \alpha_v \,(x_t y_u - x_u y_t)$$

$$= \underline{\frac{\partial F_x}{\partial x} x_t (y_u z_v - y_v z_u)} + \frac{\partial F_x}{\partial y} y_t (y_u z_v - y_v z_u) + \frac{\partial F_x}{\partial z} z_t (y_u z_v - y_v z_u)$$

$$- \frac{\partial F_y}{\partial x} x_t (x_u z_v - x_v z_u) - \underline{\frac{\partial F_y}{\partial y} y_t (x_u z_v - x_v z_u)} - \frac{\partial F_y}{\partial z} z_t (x_u z_v - x_v z_u)$$

$$+ \frac{\partial F_z}{\partial x} x_t (x_u y_v - x_v y_u) + \frac{\partial F_z}{\partial y} y_t (x_u y_v - x_v y_u) + \underline{\frac{\partial F_z}{\partial z} z_t (x_u y_v - x_v y_u)}$$

$$+ \underline{\frac{\partial F_x}{\partial x} x_u (y_v z_t - y_t z_v)} + \frac{\partial F_x}{\partial y} y_u (y_v z_t - y_t z_v) + \frac{\partial F_x}{\partial z} z_u (y_v z_t - y_t z_v)$$

$$- \frac{\partial F_y}{\partial x} x_u (x_v z_t - x_t z_v) - \underline{\frac{\partial F_y}{\partial y} y_u (x_v z_t - x_t z_v)} - \frac{\partial F_y}{\partial z} z_u (x_v z_t - x_t z_v)$$

$$+ \frac{\partial F_z}{\partial x_u} x_u (x_v y_t - x_t y_v) + \frac{\partial F_z}{\partial y} y_u (x_v y_t - x_t y_v) + \underline{\frac{\partial F_z}{\partial z} z_u (x_v y_t - x_t y_v)}$$

$$+ \underline{\frac{\partial F_x}{\partial x} x_v (y_t z_u - y_u z_t)} + \frac{\partial F_x}{\partial y} y_v (y_t z_u - y_u z_t) + \frac{\partial F_x}{\partial z} z_v (y_t z_u - y_u z_t)$$

$$- \frac{\partial F_y}{\partial x} x_v (x_t z_u - x_u z_t) - \underline{\frac{\partial F_y}{\partial y} y_v (x_t z_u - x_u z_t)} + \frac{\partial F_y}{\partial z} z_v (x_t z_u - x_u z_t)$$

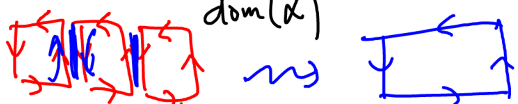
$$+ \frac{\partial F_z}{\partial x} x_v (x_t y_u - x_u y_t) + \frac{\partial F_z}{\partial y} y_v (x_t y_u - x_u y_t) + \underline{\frac{\partial F_z}{\partial z} z_u (x_t y_u - x_u y_t)}$$

$$= \frac{\partial F_x}{\partial x}\begin{vmatrix} x_t & x_u & x_v \\ y_t & y_u & y_v \\ z_t & z_u & z_v \end{vmatrix} + \frac{\partial F_y}{\partial y}\begin{vmatrix} x_t & x_u & x_v \\ y_t & y_u & y_v \\ z_t & z_u & z_v \end{vmatrix} + \frac{\partial F_z}{\partial z}\begin{vmatrix} x_t & x_u & x_v \\ y_t & y_u & y_v \\ z_t & z_u & z_v \end{vmatrix}$$

$$= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\begin{vmatrix} x_t & x_u & x_v \\ y_t & y_u & y_v \\ z_t & z_u & z_v \end{vmatrix} = \operatorname{div}(\vec{F})\,|\partial\alpha|$$

Hence we conclude the result from the following calculation:

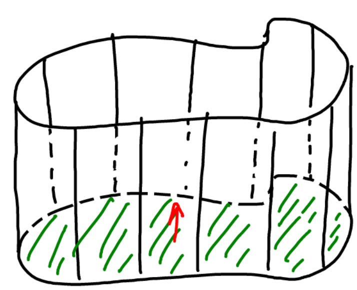
$$\iint_{\partial R} \vec{F}\cdot d\vec{S} = \iint_{\partial\,\mathrm{dom}(\alpha)} \vec{G}\cdot d\vec{S} = \iiint_{\mathrm{dom}(\alpha)} \operatorname{div}(\vec{G})\,dV = \iiint_R \operatorname{div}(\vec{F})\,dV \quad \blacksquare$$



NB: Our proof implies the general result. Every compact region $R \subseteq \mathbb{R}^3$ is a finite union of regions parameterized by rectangles. By arranging carefully, we can sum over those regions to obtain the desired integral formula ☺.

We now prove Green's Theorem via the Divergence Theorem.

Letting $R = D \times [0,1]$, we calculate:



$$\int_{\partial D} \langle P,Q\rangle\cdot\vec{N}\,ds = \iint_{\partial R} \langle P,Q,0\rangle\cdot d\vec{S}$$

$P = P(x,y)$, $Q = Q(x,y)$

$$\overset{\text{div thm}}{=} \iiint_R \operatorname{div}\langle P,Q,0\rangle\,dV$$

$$= \iint_D \operatorname{div}\langle P,Q\rangle \int_{z=0}^{1} dz\,dA = \iint_D \operatorname{div}\langle P,Q\rangle\,dA$$

Ex: Verify the Divergence Theorem for $\vec{F} = \langle y^2 z^3, 2yz, 4z^2 \rangle$ and R the unit ball centered at the origin.

Sol: Need to verify $\displaystyle\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$

__Compute the Surface Integral:__

R is parametrized by
$$\begin{cases} x = \rho \sin(\varphi)\cos(\theta) \\ y = \rho \sin(\varphi)\sin(\theta) \\ z = \rho\cos(\varphi) \end{cases}$$
on $(\rho, \theta, \varphi) \in [0,1] \times [0, 2\pi] \times [0, \frac{\pi}{2}]$.

$\therefore\ \partial R$ is parameterized by $\vec{S}(\theta, \varphi) = \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi) \rangle$

$$\vec{F}(\vec{S}(\theta,\varphi)) = \langle \sin^2(\theta)\sin^2(\varphi)\underline{\cos^3(\varphi)},\ 2\sin(\varphi)\sin(\theta)\underline{\cos(\varphi)},\ 4\underline{\cos^2(\varphi)} \rangle$$

$$-\vec{S}_\theta \times \vec{S}_\varphi = -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) & 0 \\ \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & -\sin(\varphi) \end{vmatrix}$$

inward normal, so negate

$$= -\left\langle -\sin^2(\varphi)\cos(\theta),\ -\sin^2(\varphi)\sin(\theta),\ -\sin(\varphi)\cos(\varphi)\left(\sin^2(\theta) + \cos^2(\theta)\right) \right\rangle$$

$$= +\sin(\varphi)\ \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi) \rangle$$

$$\iint_{\partial R} \vec{F} \cdot d\vec{S}$$

$$= -\iint_D \sin(\varphi)\cos(\varphi)\left(\sin^3(\varphi)\cos^2(\varphi)\sin^2(\theta)\cos(\theta) + 2\sin^2(\varphi)\sin^2(\theta) + 4\cos^2(\varphi)\right) dA$$

$$= -\int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi} \sin(\varphi)\cos(\varphi)\left(\sin^3(\varphi)\cos^2(\varphi)\sin^2(\theta)\cos(\theta) + \sin^2(\varphi)\left(1 - \cos(2\theta)\right) + 4\cos^2(\varphi)\right) d\theta\, d\varphi$$

$$= -\int_{\varphi=0}^{\pi} \sin(\varphi)\cos(\varphi)\left[\tfrac{1}{3}\sin^3(\varphi)\cos^2(\varphi)\sin^3(\theta) + \sin^2(\varphi)\left(\theta - \tfrac{1}{2}\sin(2\theta)\right) + 4\theta\cos^2(\varphi)\right]_{\theta=0}^{2\pi} d\varphi$$

$$= -\int_{\varphi=0}^{\pi} \sin(\varphi)\cos(\varphi)\Big(\big((0-0)\big) + \sin^2(\varphi)\big((2\pi-0)-(0-0)\big) + 4(2\pi-0)\cos^2(\varphi)\Big)\,d\varphi$$

$$= -\int_{\varphi=0}^{\pi} \cos(\varphi)\sin(\varphi)\Big(2\pi\big(\sin^2(\varphi) + 4\cos^2(\varphi)\big)\Big)\,d\varphi$$

$$= 2\pi\int_{\varphi=0}^{\pi} \big(\cos(\varphi) + 3\cos^3(\varphi)\big)\cdot -\sin(\varphi)\,d\varphi \qquad \left(\begin{array}{ll} u = \cos(\varphi) & u(\pi) = -1 \\ du = -\sin(\varphi)d\varphi & u(0) = 1 \end{array}\right.$$

$$= 2\pi\int_{u=1}^{-1} (u + 3u^3)\,du = 2\pi\left[\frac{1}{2}u^2 + \frac{3}{4}u^4\right]_{u=1}^{-1}$$

$$= \pi\left(\left(1 + \frac{3}{2}\right) - \left(1 + \frac{3}{2}\right)\right) = 0$$

$y^2z^3 \quad 2yz \quad 4z^2$

Compute the Volume integral:

$$\iiint_R \operatorname{div}(\vec{F})\,dV$$

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}[y^2z^3] + \frac{\partial}{\partial y}[2yz] + \frac{\partial}{\partial z}[4z^2]$$

$$= 0 + 2z + 8z = 10z$$

$$\therefore \iiint_R \operatorname{div}(\vec{F})\,dV = \iiint_R 10z\,dV_{xyz}$$

$$= \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} 10\rho\cos(\varphi)\,\rho^2\sin(\varphi)\,d\varphi\,d\theta\,d\rho$$

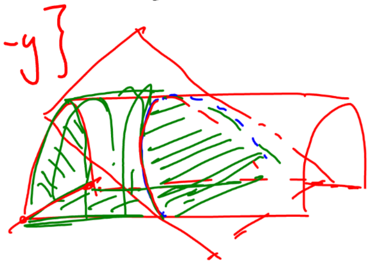
$$= 10\int_{\rho=0}^{1}\rho^3\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi}\cos(\varphi)\sin(\varphi)\,d\varphi\,d\theta\,d\rho$$

$$= 5\int_{\rho=0}^{1}\rho^3\int_{\theta=0}^{2\pi}\big[\sin^2(\varphi)\big]_{\varphi=0}^{\pi}\,d\theta\,d\rho$$

$$= 5\int_{\rho=0}^{1}\rho^3\int_{\theta=0}^{2\pi} 0\,d\theta\,d\rho = 0 \qquad \blacksquare$$

Ex: Compute $\iint_{\partial R} \vec{F} \cdot d\vec{S}$ for $\vec{F} = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$ for R bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, and $y + z = 2$.

Sol: $R = \{(x,y,z) : -1 \le x \le 1,\ 0 \le y \le x^2+1,\ 0 \le z \le 2-y\}$



$1 - x^2 = z = 0$ i.e. $x^2 = 1$ i.e. $x = \pm 1$

$y + 1 - x^2 = 2$ i.e. $y = 1 + x^2$

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}\left[y^2 + e^{xz^2}\right] + \frac{\partial}{\partial z}[\sin(xy)]$$

$$= y + 2y + 0 = 3y$$

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$$

$$= \int_{x=-1}^{1} \int_{y=0}^{x^2+1} \int_{z=0}^{2-y} 3y \, dz\, dy\, dx$$

$$= \int_{x=-1}^{1} \int_{y=0}^{x^2+1} 3y\,[z]_{z=0}^{2-y}\, dy\, dx$$

$$= \int_{x=-1}^{1} \int_{y=0}^{x^2+1} (6y - 3y^2)\, dy\, dx$$

$$= \int_{x=-1}^{1} \left[3y^2 - y^3\right]_{y=0}^{x^2+1} dx = \int_{x=-1}^{1} \left(3(x^2+1)^2 - (x^2+1)^3 - 0\right) dx$$

$$= \int_{x=-1}^{1} (x^2+1)^2 (3 - x^2 - 1)\, dx$$

$$= \int_{x=-1}^{1} (x^4 + 2x^2 + 1)(2 - x^2)\, dx$$

$$= \int_{x=-1}^{1} (2x^4 + 4x^2 + 2 - x^6 - 2x^4 - x^2)\, dx = \int_{x=-1}^{1} (2 + 3x^2 - x^6)\, dx$$

$$= \left[2x + x^3 - \tfrac{1}{7}x^7\right]_{x=-1}^{1} = 2(1-(-1)) + (1-(-1)) - \tfrac{1}{7}(1-(-1))$$

$$= 2(2) + 2 - \tfrac{1}{7}(2) = 2\left(2 + 1 - \tfrac{1}{7}\right) = 2\left(\tfrac{21-1}{7}\right) = \tfrac{40}{7}$$

Ex: Compute the flux of $\vec{F} = \langle z, y, x \rangle$ across $x^2+y^2+z^2=1$.

Sol: $\iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV = \iiint_R (0+1+0)\, dV$

$= \iiint_R 1\, dV = \text{Vol(unit sphere)} = \frac{4}{3}\pi \cdot 1^3 = \frac{4}{3}\pi$ ▨

Ex: Calculate the flux of $\vec{F} = \langle xe^y, z-e^y, -xy \rangle$ across the ellipsoid R with equation $x^2+2y^2+3z^2=4$.

Sol: $\iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV = \iiint_R (e^y + -e^y + 0)\, dV = \iiint_R 0\, dV = 0$ ▨

Ex: Calculate the flux of $\vec{F} = \langle x^2yz, xy^2z, xyz^2 \rangle$ across the boundary of $R = [0,a]\times[0,b]\times[0,c]$ for $a,b,c>0$.

Sol: $\iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV = \iiint_R (2xyz + 2xyz + 2xyz)\, dV$

$= \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} 6xyz\, dz\, dy\, dx = 3\int_{x=0}\int_{y=0}^{b}\left[xyz^2\right]_{z=0}^{c} dy\, dx$

$= 3\int_{x=0}^{a} \frac{1}{2}\left[xy^2\right]_{y=0}^{b}(c^2-0)\, dx = \frac{3}{2}c^2\left[\frac{1}{2}x^2\right]_{x=0}^{a}(b^2-0)$

$= \frac{3}{4}b^2c^2(a^2-0) = \frac{3}{4}a^2b^2z^2$ ▨

Ex: Compute $\iint_S \langle 3x, xy, 2xz \rangle \cdot d\vec{S}$ for $S = \partial([0,1]^3)$.

Sol: Exercise for you to test your skills!! ▨